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D-brane Configurations and Nicolai Map in Supersymmetric Yang-Mills Theory

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Abstract

We discuss some properties of a supersymmetric matrix model that is the dimensional reduction of supersymmetric Yang-Mills theory in ten dimensions and which has been recently argued to represent the short-distance structure of M theory in the infinite momentum frame. We describe a reduced version of the matrix quantum mechanics and derive the Nicolai map of the simplified supersymmetric matrix model. We use this to argue that there are no phase transitions in the large-N limit, and hence that S-duality is preserved in the full eleven dimensional theory.

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The conventional understanding of the spacetime structure of string theory has drastically changed over the last few years. It has been realized recently that all ten dimensional superstring theories are related by non-perturbative dualities and that they can be thought of as originating, via Kaluza-Klein types of compactifications, from a single, eleven dimensional theory known as 'M theory' (see [1] for recent reviews). The dynamics of this theory are not yet fully understood. Some of the central objects in understanding the string dualities are non-perturbative, p-dimensional degrees of freedom known as Dp-branes [2], on which the endpoints of strings can attach (with Dirichlet boundary conditions). The low-energy dynamics of a system of N parallel D-branes can be described by an $N\times N$ matrix model obtained from the dimensional reduction of ten dimensional supersymmetric Yang-Mills theory with gauge group $U(N)$ [3, 4]. The large-N limit of this matrix model has been recently conjectured to describe the small distance spacetime structure of M theory in the infinite momentum frame [5]. The explicit solution of this matrix model therefore has the potential of providing a non-perturbative description of the largely unknown dynamical objects describing the short distance behaviour of the full eleven dimensional theory [6].

In this Letter we will discuss some basic properties of the supersymmetric matrix model introduced in [4, 5]. We examine a particular reduction of the model proposed in [4, 5] to static D-brane configurations with transverse $SO(8)$ rotational symmetry. We explicitly construct the Nicolai map associated with the supersymmetry in this reduced model and use it to analyse Schwinger-Dyson equations of the matrix model. We show that the results from this analysis are consistent with other known results of the D-brane field theory, and also that this simple approach gives some insights into the structure of the full matrix model. In particular, the reduced model seems to have no phase transitions in the large-N limit and S-duality is preserved in this representation of the full eleven-dimensional theory.

First, we discuss some aspects of the representation of systems of D-branes by Yang-Mills fields. Consider the gauged supersymmetric matrix quantum mechanics with action [4, 5]

$$S=\frac{1}{2g}\int dt~\mathrm{tr}\left(\sum_{i=1}^{9}\Big(D_tX^i\Big)^2-\sum_{i<j}\Big[X^i,X^j\Big]^2+2\psi^\alpha D_t\psi_\alpha-2\sum_{i=1}^{9}\psi^\alpha(\gamma_i)_\alpha^\beta\Big[\psi_\beta,X^i\Big]\right) \tag{1}$$

where $D_tY=\partial_tY-i[A_0(t),Y]$ is the temporal component of the gauge covariant derivative, the trace is taken over unitary group indices, and we have chosen units in which the string tension is $\alpha'=1/2\pi$. Here $X^i(t)=[X^i_{ab}(t)]$, $a,b=1,\ldots,N$, $i=1,\ldots,9$, are $N\times N$ Hermitian matrices in the adjoint representation of $U(N)$ which are obtained as the spatial components in the reduction to $(0+1)$ dimensions of a $(9+1)$ dimensional $U(N)$ Yang-Mills field $A_\mu(x,t)$, $\mu=0,\ldots,9$. They describe the collective coordinates of a system of N parallel D0-branes (with infinitesimal separation), and they transform under the vector representation of the rotation group $SO(9)$ of the space transverse to the compactified eleventh dimension of

the underlying supergravity theory. The superpartners of the matrices X^i are the Majorana spinors $\psi^\alpha(t) = [\psi^\alpha_{ab}(t)]$, $\alpha = 1, \ldots, 16$, which transform under the 16-dimensional spinor representation of $SO(9)$, and under the adjoint representation of the gauge group $U(N)$. The Dirac matrices γ_i are the generators of the spin(9) Clifford algebra

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \tag{2}$$

in a Majorana basis. The coupling constant g is related to the eleven dimensional compactification radius R by

$$R = g^{2/3} l_P \tag{3}$$

where l_P is the eleven dimensional Planck length.

The action (1) describes the short-distance properties of D0-branes in weakly-coupled type-IIA superstring theory [4, 6]. It was argued in [5] to be the most general infinite momentum frame action with at most two derivatives which is invariant under the $U(N)$ gauge group and the full eleven dimensional Lorentz group. It is further invariant under the infinitesimal $\mathcal{N} = 1$ supersymmetry transformation

$$\begin{aligned} \delta_\varepsilon X^i &= -2\varepsilon^\alpha (\gamma^i)_\alpha^\beta \psi_\beta \\ \delta_\varepsilon \psi^\alpha &= \frac{1}{2}\left(\sum_i D_t X^i (\gamma_i)^\alpha_\beta + \frac{1}{2}\sum_{i<j}\left[X^i, X^j\right][\gamma_i, \gamma_j]^\alpha_\beta\right)\varepsilon^\beta \\ \delta_\varepsilon A_0 &= -2\varepsilon^\alpha \psi_\alpha \end{aligned} \tag{4}$$

where ε^α are 16 global Majorana spinor parameters. There are another set of 16 supersymmetries which are realized trivially as $\delta_{\varepsilon'}\psi^\alpha = \varepsilon'^\alpha$, $\delta_{\varepsilon'} X^i = \delta_{\varepsilon'} A_0 = 0$, where ε and ε' are independent supersymmetry parameters. Together, these two sets of supersymmetry transformations (with slight modifications) generate the full 32-dimensional $\mathbf{16} \oplus \mathbf{16}$ representation of the super-Galilean group of the eleven dimensional theory in the light-cone frame. A number of properties of M theory have been verified using the action (1) [5],[7]–[9].

It may seem puzzling at first sight that a dimensionally-reduced gauge field A yields the appropriate D-brane coordinatization. What is intriguing though is that string theoretic T-duality is a key to this connection. If we compactify the first spatial dimension onto a circle $S^1_{\mathcal{R}}$ of radius $\mathcal{R}$ (or more generally several dimensions onto a torus), then the angular coordinate x^1 (or coordinates x^i) takes values in the interval $x^1 \in [0, 2\pi\mathcal{R}]$. The original supersymmetric Yang-Mills theory in ten dimensions describes the low-energy sector of open superstrings, and in this theory there exists other gauge-invariant variables, namely the path-ordered Wilson loop operators

$$W[A] = \operatorname{tr} P \exp\left(i \oint_{S^1_{\mathcal{R}}} A_i \, dx^i\right) \tag{5}$$

which are invariant under the large gauge transformations which wind around the compactified direction. The connection appearing in the argument of the exponential in (5) lies in the adjoint representation of $U(N)$. The corresponding classical gauge field orbits are topological and lie in the interval $A_1 \in [0, \frac{2\pi}{\mathcal{R}}]$. For each coordinate x^i living on a circle $S^1_{\mathcal{R}_i}$ there is a "dual" coordinate $X^i = \alpha' A_i$ which also lives on a circle $S^1_{r_i}$ but with a dual radius $r_i = \alpha'/\mathcal{R}_i$. As we have mentioned the ten-dimensional supersymmetric Yang-Mills theory describes the low-energy dynamics of open strings with Neumann boundary conditions. Under T-duality $\mathcal{R} \to \alpha'/\mathcal{R}$ the Neumann boundary conditions are transcribed into Dirichlet boundary conditions. But this means that the topological gluon degrees of freedom are converted into the D-brane fields $X(t) = \alpha' A(t)$ describing open strings with Dirichlet boundary conditions. Thus the ten-dimensional gluon fields describe the dual theory of the D-branes and T-duality naturally identifies the topological orbits of the Yang-Mills fields as the D-brane coordinates. The role of T-duality in the matrix model (1) has also been addressed from other points of view in [8].

A question which now arises is how to really measure the original angular coordinate x^i given the gauge field coordinate X^i. The solution is to consider more complicated objects, such as the abelian Wilson loops with a winding number $n \in \mathbb{Z}$

$$W_n[A] = \exp\left(in \oint_{S^1_{\mathcal{R}}} A_i \, dx^i\right) \tag{6}$$

which are associated with a single D-brane configuration. In the general case, i.e. in the case of a non-abelian gauge group, we should consider Wilson loops in different representations of $U(N)$, but for the sake of illustration we shall discuss here only the $U(1)$ Wilson loops (6). This simplication can be thought of as a compactification of all of the ten dimensions on which the group variables (5) are restricted to the maximal torus of the $U(N)$ gauge group. Using these objects and T-duality one can in principle obtain the angular coordinate x^i through a superposition $\sum_{n\in\mathbb{Z}} W_n[A]$ of different Wilson loops (using harmonic analysis on unitary groups in the general case).

In fact, this construction demonstrates that the Wilson loops in the "N-theory" (i.e. ordinary open strings with Neumann boundary conditions) are equivalent to the vertex operators in the "D-theory" (open strings with Dirichlet boundary conditions). To see this, we expand the topological gauge field configurations of the low-energy description of strings on the compactified space in Fourier modes as

$$A_1(t,x) = \sum_{\ell\in\mathbb{Z}} A_1^{(\ell)}(t)\, \mathrm{e}^{i\ell\mathcal{R}x} \qquad , \qquad x \in [0, 2\pi/\mathcal{R}] \tag{7}$$

These dual fields are described by a $(1+1)$ dimensional gauge theory that is also dimensionally reduced from the ten-dimensional supersymmetric Yang-Mills theory [8]. Then the Wilson loop (6) becomes

$$W_n = \mathrm{e}^{in\mathcal{R}A_1^{(0)}(t)} = \mathrm{e}^{ip_n X^1(t)} \tag{8}$$

where $p_n = n/r = n\mathcal{R}/\alpha'$ is the momentum of the string winding mode in the compactified direction. Thus T-duality converts the non-commuting position matrices X^i describing the D-brane configurations into gauge fields in the dual theory, and also the winding number of the topological gauge field modes into the string momentum in the compactified direction. In the general case then, we can conjecture the equivalence between the string scattering amplitudes of the D-theory defined by correlators of the vertex operators and expectation values of the Wilson loop operators in the dual N-theory[2],

$$\langle W_{\vec{n}_1}[A] \cdots W_{\vec{n}_k}[A] \rangle_N = \left\langle \mathrm{e}^{i\vec{n}_1 \cdot \vec{X}} \cdots \mathrm{e}^{i\vec{n}_k \cdot \vec{X}} \right\rangle_D \tag{9}$$

where $\vec{n}_j \in \mathbb{Z}^9$ and we have defined $W_{\vec{n}_j}[A] = \prod_{i=1}^9 W_{(\vec{n}_j)^i}[A]$ in terms of the abelian Wilson loops (6). In the non-abelian case, the correlators (9) will generalize in the appropriate way in terms of representations of $U(N)$.

We now examine the problem of obtaining an explicit solution of the matrix model (1). For this, we further dimensionally reduce the theory described by (1) to a zero-dimensional $N \times N$ supersymmetric matrix model, i.e. we ignore the time dependence in (1) and work in the Weyl gauge $A_0 = 0$. This means that we are studying the model separately over each constant time slice describing a static configuration of the D0-branes. This reduction can be thought of as originating by compactifying the time direction of the ten dimensional Yang-Mills theory where the adjoint representation fermions have periodic boundary conditions, and then taking the limit in which the radius of compactification vanishes. This simplification has the advantage of eliminating non-local operators that would appear from the time-dependence. We shall discuss the inclusion of time-dependent fields at the end of this Letter.

With this further reduction, the partition function of the model is given by the finite-dimensional matrix integral

$$Z = \int \prod_{a,b} \prod_i dX^i_{ab} \prod_\alpha d\psi^\alpha_{ab} \, \exp\left\{ \frac{N}{2g} \, \mathrm{tr} \left(\sum_{i<j} \left[X^i, X^j\right]^2 - 2 \sum_i \psi^\alpha (\gamma_i)_\alpha^\beta \left[\psi_\beta, X^i\right] \right) \right\} \tag{10}$$

We can expand the matrix integration variables in (10) in a basis T^A of the unitary group as $X^i = X^i_A T^A$ and $\psi^\alpha = \psi^\alpha_A T^A$, where $A = 1, \ldots, N^2$ and the Hermitian $U(N)$ generators satisfy

$$\left[T^A, T^B\right] = i f^{AB}{}_C T^C \qquad , \qquad \mathrm{tr}\; T^A T^B = \frac{1}{2} \delta^{AB} \tag{11}$$

The integration over the Majorana fermions in (10) is Gaussian and can be evaluated explicitly using the Berezin integration rules for ψ^α_A. It produces a square root of the determinant determined by the representation of the adjoint action of X^i, and (10) becomes

$$Z = c_N \int \prod_i \prod_{D=1}^{N^2} dX^i_D \; \mathrm{Pfaff} \left[\frac{i}{2g} f^{ABC} \sum_i (\gamma_i)_\alpha^\beta X^i_C \right] \exp\left\{ \frac{N}{2g} \sum_{i<j} \mathrm{tr} \left[X^i, X^j\right]^2 \right\} \tag{12}$$

[2]See [10] for another discussion of the relationship between string vertex operators and Wilson lines.

where the Pfaffian is taken over both the adjoint $U(N)$ representation indices $A, B = 1, \ldots, N^2$ and the spin(9) indices $\alpha, \beta = 1, \ldots, 16$. Here and in the following c_N denotes an irrelevant numerical constant.

We would now like to exploit the supersymmetry (4) of the zero dimensional model to compute correlation functions of the matrix model. When the number of bosonic and fermionic degrees of freedom are the same, the supersymmetry is maximal, in that it holds even when the fields are off-shell. In the present model, the number of fermionic and bosonic degrees of freedom do not match. Normally, supersymmetry would require on-shell fields and the addition of auxilliary fields to make the number of physical boson and fermion modes equal. However, we can adjust things to match by exploiting the original interpretation of the matrices X^i from the dimensional reduction of ten-dimensional supersymmetric Yang-Mills theory. The latter theory can be gauge-fixed and quantized in the light-cone gauge, after which there are only eight propagating gluon degrees of freedom corresponding to the various possible transverse polarizations. Since a Majorana-Weyl spinor in ten dimensions has eight physical modes, the minimal Yang-Mills action in ten dimensions is supersymmetric without the need of introducing auxilliary fields. Thus we match the collective degrees of freedom X^i of the system of D-branes with the physical modes of the full Yang-Mills theory by reducing the target space degrees of freedom from nine to eight by setting $X^9 = 0$ and working in the Majorana-Weyl representation of spin(9) in the matrix model above. The constraint $X^9 = 0$ can be thought of as a light-cone gauge fixing condition in the nine-dimensional transverse space. Although not precise from the point of view of the M theory dynamics, this simplification produces a toy model that will shed light on some of the properties of the nine-dimensional theory that we started with[3].

With this simplification we now exploit some features of the group theory for $SO(9)$. The Dirac generators of spin(9) in the Majorana-Weyl basis can be constructed from the reducible $\mathbf{8}_s \oplus \mathbf{8}_c$ chiral representation of spin(8) by decomposing the 16-dimensional gamma-matrices in the 8×8 block form

$$\gamma_i = \begin{pmatrix} 0 & (\gamma_i)_\alpha^{\dot{\alpha}} \\ (\gamma_i)_{\dot{\beta}}^{\beta} & 0 \end{pmatrix} \quad , \qquad i = 1, \ldots, 8 \tag{13}$$

where $(\gamma_i)_{\dot{\alpha}}^{\alpha} = (\gamma_i^T)_\alpha^{\dot{\alpha}}$, $\alpha, \dot{\alpha} = 1, \ldots, 8$, are the Dirac generators of spin(8). The Clifford algebra (2) is then equivalent to the equations

$$(\gamma_i)_\alpha^{\dot{\alpha}} (\gamma_j)_{\dot{\alpha}}^{\beta} + (\gamma_j)_\alpha^{\dot{\alpha}} (\gamma_i)_{\dot{\alpha}}^{\beta} = 2\delta_{ij} \delta_\alpha^\beta \tag{14}$$

[3]Some different reductions of the matrix quantum mechanics (1) have also been suggested. In [11] it was argued that the model can be truncated to zero dimensions by augmenting the transverse rotational symmetry to $SO(11)$ and viewing the matrix model as the dimensional reduction of supersymmetric Yang-Mills theory in $(10 + 2)$ dimensions. In [10] it was argued that the D-brane field theory associated with weakly-coupled type-IIB superstrings could be viewed as the large-N reduction of the ten dimensional supersymmetric Yang-Mills theory.

and similarly with dotted and undotted chiral indices interchanged. The spin(8) Dirac generators can be expressed explicitly as direct products of 2×2 block matrices by

$$
\begin{aligned}
(\gamma_1)_\alpha^{\dot\alpha} &= -i\sigma_2\otimes\sigma_2\otimes\sigma_2 \quad &,\quad (\gamma_2)_\alpha^{\dot\alpha} &= i\mathbf{1}\otimes\sigma_1\otimes\sigma_2\\
(\gamma_3)_\alpha^{\dot\alpha} &= i\mathbf{1}\otimes\sigma_3\otimes\sigma_2 \quad &,\quad (\gamma_4)_\alpha^{\dot\alpha} &= i\sigma_1\otimes\sigma_2\otimes\mathbf{1}\\
(\gamma_5)_\alpha^{\dot\alpha} &= i\sigma_3\otimes\sigma_2\otimes\mathbf{1} \quad &,\quad (\gamma_6)_\alpha^{\dot\alpha} &= i\sigma_2\otimes\mathbf{1}\otimes\sigma_1\\
(\gamma_7)_\alpha^{\dot\alpha} &= i\sigma_2\otimes\mathbf{1}\otimes\sigma_3 \quad &,\quad (\gamma_8)_\alpha^{\dot\alpha} &= \mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}
\end{aligned}
\tag{15}
$$

where $\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2=\begin{pmatrix}0&i\\-i&0\end{pmatrix}$ and $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ are the Pauli spin matrices. The remaining spin(9) Dirac matrix is then $\gamma_9=\gamma_1\gamma_2\cdots\gamma_8=\begin{pmatrix}\mathbf{1}&0\\0&-\mathbf{1}\end{pmatrix}$.

The block decomposition (13) of the first eight gamma-matrices shows that the Pfaffian in (12) with $X^9=0$ becomes squared, with the spinor part of the determinant restricted to the spin(8) chiral indices. Then we can write the partition function as

$$
Z=c_N\int\prod_{i=1}^{8}\prod_{D=1}^{N^2}dX_D^i\ \det_{A,B;1\le\alpha,\dot\alpha\le 8}\left[\frac{1}{2g}f^{ABC}\sum_{i=1}^{8}(\gamma_i)_\alpha^{\dot\alpha}X_C^i\right]\exp\left\{\frac{N}{2g}\sum_{1\le i<j\le 8}\mathrm{tr}\left[X^i,X^j\right]^2\right\} \tag{16}
$$

This effective reduction to eight dimensions matches bosonic and fermionic degrees of freedom and allows us to exploit the supersymmetry in a simple way to completely solve the matrix model. Essentially it enables us to use the triality property of the eight-dimensional rotation group, i.e. that there exists automorphisms between the vector and chiral spinor representations of $SO(8)$.

We now label the eight spatial indices i as the chiral indices $\alpha,\dot\alpha$ of the spinor representation, and the first eight components of the 16-component spinor field ψ as the chiral parts ψ^α and the last eight components as the anti-chiral parts $\psi^{\dot\alpha}$ in the $\mathbf{8}_s\oplus\mathbf{8}_c$ decomposition of spin(9) above. The static reduction of the quantum mechanical action (1) can then be written in the standard form of an $\mathcal{N}=1$ supersymmetric field theory (after integration over superspace coordinates) as

$$
S_0=-\frac{1}{2g}\left[\sum_{i,A}\left(\frac{\partial F}{\partial X_A^i}\right)^2+\psi_A^\alpha\left(\frac{\partial^2F}{\partial X_A^\alpha\partial X_B^{\dot\alpha}}\right)\psi_B^{\dot\alpha}+\psi_A^{\dot\alpha}\left(\frac{\partial^2F}{\partial X_A^{\dot\alpha}\partial X_B^{\alpha}}\right)\psi_B^{\alpha}\right] \tag{17}
$$

where the super-potential is

$$
F(X)\equiv\frac{1}{3}(\gamma_k)_\alpha^{\dot\alpha}\ \mathrm{tr}\ X^k\left[X^\alpha,X_{\dot\alpha}\right]=\frac{1}{6}(\gamma_k)_\alpha^{\dot\alpha}f^{AB}{}_C X_A^kX_B^\alpha X_{\dot\alpha}^C \tag{18}
$$

The representation (17) can be derived using the symmetry properties of the gamma-matrices (15), the $U(N)$ Jacobi identity

$$
f^{ABC}f^{ADE}=f^{ADB}f^{ACE}-f^{ADC}f^{ABE} \tag{19}
$$

and the $SO(8)$ Fierz identity

$$(\gamma_i)^{\alpha}_{\dot\alpha}(\gamma_j)^{\dot\beta}_{\alpha} = \delta_{ij}\delta^{\dot\beta}_{\dot\alpha} + (\gamma_{ij})^{\dot\beta}_{\dot\alpha} \tag{20}$$

where we have introduced the spinor matrix

$$(\gamma_{ij})^{\beta}_{\alpha} = \frac{1}{2}\left((\gamma_i)^{\dot\alpha}_{\alpha}(\gamma_j)^{\beta}_{\dot\alpha} - (\gamma_j)^{\dot\alpha}_{\alpha}(\gamma_i)^{\beta}_{\dot\alpha}\right) \tag{21}$$

and similarly for $(\gamma_{ij})^{\dot\beta}_{\dot\alpha}$.

The form of the action (17) identifies the Nicolai map (i.e. the Hubbard-Stratonvich transformation for the bosonic potential $\sum_{i<j}\mathrm{tr}[X^i,X^j]^2$) [12] of this supersymmetric field theory as

$$W^A_k(X) \equiv \frac{\partial F}{\partial X^k_A} = \frac{1}{2}(\gamma_k)^{\dot\alpha}_{\alpha} f^{AB}{}_{C} X^{\alpha}_{B} X^{C}_{\dot\alpha} \qquad \text{or} \qquad W^{ab}_k = \frac{1}{2}(\gamma_k)^{\dot\alpha}_{\alpha}[X^\alpha, X_{\dot\alpha}]^{ab} \tag{22}$$

From (17) we see that the Jacobian factor $|\det[\partial W^A_k/\partial X^j_B]|^{-1}$ which arises in the change of variables $X \to W(X)$ in the partition function (16) will cancel exactly with the determinant that comes from integrating out the chiral fermion fields. The partition function is thus trivially a Gaussian Hermitian matrix integral and is formally unity,

$$Z = \frac{c_N}{(2g)^{64N^2}} \int \prod_{i=1}^{8}\prod_{A=1}^{N^2} dW^A_i \;\; \mathrm{e}^{-\frac{N}{4g}(W^A_i)^2} = c_N \tag{23}$$

The free energy $\log Z$ is thus trivially an analytic function of the coupling constant g everywhere and it does not exhibit any phase transitions, even in the large-N limit. Furthermore, the correlation functions which are invariant under the supersymmetry transformations (4) can be obtained by differentiating the free energy with respect to the coupling constants of the model (in an appropriate superspace formulation). Thus any supersymmetric correlator of the model vanishes, which is just the standard non-renormalization that usually occurs in supersymmetric field theories. The existence of the Nicolai map and these implied properties of the matrix model are essentially the content of the supersymmetric Ward identities.

The only observables of the matrix model which are non-trivial are those which are not supersymmetric. To examine such correlation functions, we use the Nicolai map (22) to express correlators $\langle\cdot\rangle$ of the original matrix model (normalized so that $Z=1$) as free Gaussian averages $\langle\langle\cdot\rangle\rangle$ of the Nicolai field. For instance, from $\langle\langle W^{ab}_i\rangle\rangle = 0$ we deduce $\langle[X^i,X^j]^{ab}\rangle = 0$. This means that the classical ground state of the model (the minimum of the bosonic potential $\sum_{i<j}\mathrm{tr}[X^i,X^j]^2$) is that wherein the D-brane coordinates commute and have simultaneous eigenvalues corresponding to definite D0-brane positions. The full matrix model, which incorporates quantum fluctuations about the classical ground state, thus describes smeared-out D0-brane configurations in a spacetime with a non-commutative geometry [4, 5, 8].

More generally, we note that the Nicolai map $X \to W(X)$ is many-to-one, so that general correlators of the X matrices can have a multi-valued branch cut structure. To see if this is the case, we use the Nicolai field to write down a set of Schwinger-Dyson equations for the matrix model. The basic identity follows from the formula for Gaussian averages of products of even numbers of the fields W_i^{ab},

$$\left\langle\!\!\left\langle W_{i_1}^{a_1b_1} W_{i_2}^{a_2b_2} \cdots W_{i_{2n-1}}^{a_{2n-1}b_{2n-1}} W_{i_{2n}}^{a_{2n}b_{2n}} \right\rangle\!\!\right\rangle = \left(\frac{g^2}{N}\right)^n \left(\delta_{i_1i_2}\delta_{i_3i_4}\cdots\delta_{i_{2n-1}i_{2n}} + \Pi^{(i)}[i_1, i_2, \ldots, i_{2n}]\right)$$
$$\times \left(\delta_{a_1b_2}\delta_{a_2b_1}\delta_{a_3b_4}\delta_{a_4b_3}\cdots\delta_{a_{2n}b_{2n-1}} + \Pi^{(a,b)}[a_1, b_1; a_2, b_2; \ldots; a_{2n}, b_{2n}]\right) \tag{24}$$

where Π contains the sum of delta-functions over all permutations of indices. The delta-functions in the indices i_k come from the $SO(8)$ invariance of the reduced matrix model, while those in the indices a_k, b_k arise from $U(N)$ invariance. The non-vanishing correlation functions of the model are those which respect both of these symmetries.

As an explicit example, we set $n = 2$ in (24) and sum over $i_1 = i_2$, $i_3 = i_4$ and $a_1 = b_2$, $a_2 = b_1$, $a_3 = b_4$, $a_4 = b_3$ to get

$$\left\langle \frac{\mathrm{tr}}{N} (\gamma_i)_\alpha^{\dot\alpha} \left[X^\alpha, X_{\dot\alpha}\right] (\gamma^i)_\beta^{\dot\beta} \left[X^\beta, X_{\dot\beta}\right] \; \frac{\mathrm{tr}}{N} (\gamma_k)_\sigma^{\dot\sigma} \left[X^\sigma, X_{\dot\sigma}\right] (\gamma^k)_\rho^{\dot\rho} \left[X^\rho, X_{\dot\rho}\right] \right\rangle = 2^{10} g^4 \left(1 + \frac{2}{N^2}\right) \tag{25}$$

In the large-N limit, the expectation value of a product of invariant operators factorizes into a product of correlators. Thus at $N = \infty$ (25) becomes

$$\sum_{i,j} \left\langle \frac{\mathrm{tr}}{N} \left[X^i, X^j\right]^2 \right\rangle^2 = 2^{10} g^4 \tag{26}$$

On the other hand, setting $i_1 = i_3$, $i_2 = i_4$ and the a's and b's equal in the same way as above, we get the $N = \infty$ equation

$$\sum_{i,j} \left\langle \frac{\mathrm{tr}}{N} \left[X^i, X^j\right]^2 \right\rangle = 32 g^2 \tag{27}$$

Combining (26) and (27) together we find that the large-N invariant variance of the $SO(8)$ operator $\frac{\mathrm{tr}}{N}[X, X]^2$ is trivial,

$$\Delta^2 \left(\frac{\mathrm{tr}}{N}[X, X]^2\right) \equiv \sum_{i,j} \left\langle \frac{\mathrm{tr}}{N} \left[X^i, X^j\right]^2 \right\rangle^2 - \left(\sum_{i,j} \left\langle \frac{\mathrm{tr}}{N} \left[X^i, X^j\right]^2 \right\rangle\right)^2 = 0 \tag{28}$$

Note that (28) is a stronger statement than just the large-N factorization of correlators, as it implies a non-trivial factorization over the $SO(8)$ indices as well. This is one manifestation of the supersymmetry of this matrix model. Similar other such identities can be derived for higher-order correlators of the X-fields. In this formalism, it is also possible to treat n-point connected correlation functions of the model.

The forms of the correlators above (and in particular (28)) seem to suggest that all non-vanishing observables in this model are analytic functions of the coupling constant g at $N = \infty$. This in turn implies that the large-N limit of the matrix model exhibits no phase transitions as one continuously varies g. As the relationship with M theory dynamics is eventually obtained in the uncompactified limit where $R \to \infty$ [5], the Nicolai map demonstrates explicitly that this limit can be taken unambiguously since there is no variation in the analytic structure of the large-N solution. Moreover, the absence of phase transitions suggests that S-duality $g \to 1/g$ is maintained in the large-N limit of the matrix model above. A more precise examination of these properties requires the inversion of the Nicolai map (22) to get $X(W)$, which would enable one to compute arbitrary non-supersymmetric correlators of the matrix model. The problem in trying to construct this inverse map is that generally $W \neq 0$, corresponding to the fact that the D-brane coordinates live in a non-commutative spacetime, so that it is not possible to simultaneously diagonalize the X^i's and find the relationship between the eigenvalue models for the W and X fields. The entire non-triviality of the matrix model lies in the correlators of invariant combinations of the operator $X(W)$. The problem of inverting the Nicolai map has been discussed from a perturbative point of view in [13], where it was also shown that this transformation is a non-polynomial functional of the bosonic fields in the ten dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. It would be interesting to determine this inverse map, and use it to examine the properties of the Wilson loop correlators that we discussed earlier in addition to the large-N analyticity features of general correlators of the matrix model. In any case, we have formally described a method in which one can study features of the string scattering amplitudes (9).

The results described above are only precisely valid with both the elimination of the temporal dimension and the reduction to an $SO(8)$ spacetime symmetry group. If we reintroduce the time dependence of the matrix variables then the Nicolai map is determined as the non-local, time-dependent functional $\mathcal{W}^i(t) = D_t^2 X^i(t) - W^i(X(t))$ with W^i given in (22). Then the partition function yields the winding number of the multi-valued Nicolai map. The above results from the reduced matrix model, such as the analyticity in the coupling constant g, and hence the S-duality in the eleven dimensional compactification, show that the simple method discussed above has the potential of providing some insights into the structure of M theory. It would be interesting to see if the reduced matrix model can describe other features, such as membrane interactions [7], of the eleven dimensional theory. It would also be interesting to determine if the Nicolai map obtained above can be used to describe any properties of the ten dimensional supersymmetric Yang-Mills theory itself.

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References

[1] M.J. Duff, Intern. J. Mod. Phys. **A11** (1996), 5623;
J.H. Schwarz, CalTech preprint CALT-68-2065, hep-th/9607201 (1996)

[2] J. Polchinski, Phys. Rev. Lett. **75** (1995), 4724

[3] B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. **B305** [FS23] (1988), 545;
B. de Wit, M. Lüscher and H. Nicolai, Nucl. Phys. **B320** (1989), 135;
P.K. Townsend, Phys. Lett. **B373** (1996), 68

[4] E. Witten, Nucl. Phys. **B460** (1995), 335

[5] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, Rutgers preprint RU-96-95, hep-th/9610043 (1996)

[6] M.R. Douglas, D. Kabat, P. Pouliot and S.H. Shenker, Rutgers preprint RU-96-62, hep-th/9608024 (1996)

[7] M. Berkooz and M.R. Douglas, Rutgers preprint RU-96-100, hep-th/9610236 (1996);
O. Aharony and M. Berkooz, Rutgers preprint RU-96-106, hep-th/9611215 (1996);
G. Lifschytz and S.D. Mathur, Princeton preprint PUPT-1673, hep-th/9612087 (1996);
G. Lifschytz, Princeton preprint, hep-th/9612223 (1996)

[8] W. Taylor, Princeton preprint PUPT-1659, hep-th/9611042 (1996);
L. Susskind, Stanford preprint SU-ITP-96-12, hep-th/9611164 (1996);
O.J. Ganor, S. Ramgoolam and W. Taylor, Princeton preprint PUPT-1668, hep-th/9611202 (1996);
P.-M. Ho and Y.-S. Wu, Utah preprint UU-HEP/96-07, hep-th/9611233 (1996);
S. Kachru and E. Silverstein, Rutgers preprint RU-96-114, hep-th/9612162 (1996)

[9] M.R. Douglas, Rutgers preprint RU-96-111, hep-th/9612126 (1996);
M. Li, Chicago preprint EFI-96-36, hep-th/9612144 (1996);
T. Banks, N. Seiberg and S.H. Shenker, Rutgers preprint RU-96-117, hep-th/9612157 (1996)

[10] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, KEK preprint KEK-TH-503, hep-th/9612115 (1996);
M. Li, Chicago preprint EFI-96-49, hep-th/9612222 (1996);
K.-J. Hamada, KEK preprint KEK-TH-504, hep-th/9612234 (1996)

[11] V. Periwal, Princeton preprint PUPT-1665, hep-th/9611103, to appear in Phys. Rev. **D** (1997); Princeton preprint, hep-th/9612215 (1996)

[12] G. Parisi and N. Sourlas, Phys. Rev. Lett. **43** (1979), 744;
H. Nicolai, Phys. Lett. **B89** (1980), 341; Nucl. Phys. **B176** (1980), 419

[13] K. Dietz and O. Lechtenfeld, Nucl. Phys. **B255** (1985), 149;
O. Lechtenfeld, Nucl. Phys. **B274** (1986), 633